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ON THE INTERACTION OF A SMALL NUMBER OF
TRANSVERSE AND LONGITUDINAL MODES IN A PLASMA*

by

Adrian Dolinsky and Robert Goldman

University of Maryland
College Park, Maryland

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ABSTRACT

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We obtain equations for the time and space development of a small number of large amplitude transverse and longitudinal modes in a Vlasov plasma. The scale times and lengths are inversely proportional to the first power of the electric fields. For the special case of two transverse modes and a single resonantly driven longitudinal oscillation, we first classify systems according to the manner in which the size of the longitudinal field is limited, and then solve for the resulting mode evolution. As a result of the non-linear coupling, frequency conversion and polarization rotation effects appear. Qualitative differences are found for the behavior in one and two independent coordinates. In most (but not all) cases treated, the mode interactions cause changes in the electric field amplitudes rather than shifts in resonant frequencies and wave lengths.

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I. INTRODUCTION

In this paper we study the resonant coupling of a small number of transverse and longitudinal waves in a Vlasov plasma of electrons embedded in a uniformly smeared out background of positive charge.

The problem differs from earlier problems treated by the quasi-linear theory of the Vlasov equation^{1,2,3,4} due to the differences in the time scales involved. In this problem, because of the presence of only a small number of modes, the rate of change of the modes is proportional to the strength of the amplitudes, while the spatially homogeneous part of the one-particle distribution function changes at a rate proportional to the square of the amplitudes. On the other hand, in earlier problems treated by quasi-linear theory, very many modes are present with random phases. Because of the random phase approximation, both the modes and the spatially homogeneous part of the one-particle distribution function vary at rates which are proportional to the square of the amplitudes.

We obtain a set of coupled, first-order partial differential equations for the amplitudes of the modes. The equations take into account simultaneously several mechanisms which may modify the longitudinal field: linear Landau damping, detuning, and nonlinear effects in space and time due to resonant mode coupling. In addition collisional damping effects are assumed to be additive to linear Landau damping effects. We confine our analysis to situations in which the transverse fields vary through mode-mode coupling, since otherwise resonant interaction effects are weak.

Similar equations (though not based on the Vlasov equation) have been derived by several authors^{5,6,7}. However, they usually neglect one or more of the modifying mechanisms, such as the Landau or collisional damping, and/or they consider only nonlinear effects in time, leaving out nonlinear effects in space.

In section (IIA) we define a physical system whose behavior is governed by the nonlinear Vlasov equation coupled with Maxwell's equations. In section (IIB) we seek a perturbation theoretic solution of this system of equations. In section (IIC) we obtain differential equations for the amplitudes of the electric field of the plasma modes which follow from the prescription that all secular terms cancel, and in section (IID) we particularize our considerations to the case of two transverse oscillations impinging on the plasma and the resulting driven longitudinal disturbance.

Then in section (IIIA) we obtain a set of conservation equations governing the energy exchange among the modes of (IID). In (IIIB) we determine limits on the size of the longitudinal disturbance of (IID), and the corresponding length and/or time scales for change of the transverse modes due to mode-mode coupling.

After this in IIIC, IIID, and IIIE we solve the differential equations for the amplitudes explicitly for several special cases which are grouped according to the dominant limiting mechanism on the longitudinal mode.

II. PLASMA AND ELECTROMAGNETIC FIELD EQUATIONS

A. THE NONLINEAR EQUATIONS

We assume that the plasma and the electromagnetic field are describable by the Vlasov equation coupled with the Maxwell equations:

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{x}} - \frac{e}{m} (\underline{E} + \frac{1}{c} \underline{v} \times \underline{B}) \cdot \frac{\partial f}{\partial \underline{v}} = 0$$

$$\frac{\partial}{\partial \underline{x}} \cdot \underline{E} = 4\pi n_0 e (1 - \int f \, d\underline{v})$$

$$\frac{\partial}{\partial \underline{x}} \cdot \underline{B} = 0$$

$$\frac{\partial}{\partial \underline{x}} \times \underline{E} = - \frac{1}{c} \frac{\partial \underline{B}}{\partial t}$$

$$\frac{\partial}{\partial \underline{x}} \times \underline{B} = \frac{1}{c} \frac{\partial \underline{E}}{\partial t} - \frac{4\pi n_0 e}{c} \int \underline{v} f \, d\underline{v}$$

(1)

where all of the symbols have their usual meaning.

B. CHOICE OF PERTURBATION EXPANSION

We wish to solve Eqs. (1) by means of an expansion in a small parameter ϵ , where ϵ measures the departure of the system from the field-free quiescent state:

$$\epsilon \sim \frac{|f - f_0|}{f_0}, \quad (2)$$

where $f_0(\underline{v})$ is the distribution function for the quiescent state.

According to the multiple time and spatial scale scheme^{8,9} we introduce into the problem a set of time scales, t_0, t_1, t_2, \dots , and a set of spatial scales, $\underline{x}_0, \underline{x}_1, \underline{x}_2, \dots$, defined such that $t_0 = t$, $t_1 = \epsilon t$, $t_2 = \epsilon^2 t, \dots$, and $\underline{x}_0 = \underline{x}$, $\underline{x}_1 = \epsilon \underline{x}$, $\underline{x}_2 = \epsilon^2 \underline{x}, \dots$. Any function of \underline{x} and t is assumed to be a function of $\underline{x}_0, \underline{x}_1, \underline{x}_2, \dots$, t_0, t_1, t_2, \dots , respectively.

Let us seek solutions to Eqs. (1) which can be written in the form:

$$\left. \begin{aligned} f(\underline{x}, \underline{v}, t) &= f^{(0)}(\underline{v}, \underline{x}_1, t_1, \dots) + \epsilon f^{(1)}(\underline{x}_0, \underline{v}, t_0, \underline{x}_1, t_1, \dots) + \dots \\ \underline{E}(\underline{x}, t) &= \epsilon E^{(1)}(\underline{x}_0, t_0, \underline{x}_1, t_1, \dots) + \dots \\ \underline{B}(\underline{x}, t) &= \epsilon B^{(1)}(\underline{x}_0, t_0, \underline{x}_1, t_1, \dots) + \dots \end{aligned} \right\} (3)$$

The dependence of the functions on \underline{x}_0 and t_0 has to be such that the Vlasov and Maxwell equations be satisfied to orders $0(\epsilon)$, $0(\epsilon^2)$, *etc.*. The functional dependence on \underline{x}_1 , t_1 , \underline{x}_2 , t_2 , *etc.* gives us additional freedom of choice; and we can use it to eliminate secular terms from the equations to order $0(\epsilon^2)$, $0(\epsilon^3)$, *etc.*, *i.e.* to as many orders as desired.

For the sake of simplicity, let us take $f^{(0)} = f^{(0)}(|\underline{v}|, \dots)$ to be an isotropic function of velocity.

Substituting the expansion (3) into Eqs. (1), and writing all spatial and time derivatives as derivatives with respect to $\underline{x}_0, t_0, \underline{x}_1, t_1$, *etc.*, we obtain the following equations:

$$\begin{aligned} \epsilon \left[\frac{\partial f^{(1)}}{\partial t_0} + \underline{v} \cdot \frac{\partial f^{(1)}}{\partial \underline{x}_0} - \frac{e}{m} \underline{E}^{(1)} \cdot \frac{\partial f^{(0)}}{\partial \underline{v}} + \frac{\partial f^{(0)}}{\partial t_1} + \underline{v} \cdot \frac{\partial f^{(0)}}{\partial \underline{x}_1} \right] \\ + \epsilon^2 \left[\frac{\partial f^{(2)}}{\partial t_0} + \underline{v} \cdot \frac{\partial f^{(2)}}{\partial \underline{x}_0} - \frac{e}{m} \underline{E}^{(2)} \cdot \frac{\partial f^{(0)}}{\partial \underline{v}} \right. \\ \left. - \frac{e}{m} \left(\underline{E}^{(1)} + \frac{1}{c} \underline{v} \times \underline{B}^{(1)} \right) \cdot \frac{\partial f^{(1)}}{\partial \underline{v}} + \frac{\partial f^{(1)}}{\partial t_1} + \underline{v} \cdot \frac{\partial f^{(1)}}{\partial \underline{x}_1} \right. \\ \left. + \frac{\partial f^{(0)}}{\partial t_2} + \underline{v} \cdot \frac{\partial f^{(0)}}{\partial \underline{x}_2} \right] = 0, \end{aligned} \quad (4)$$

$$\epsilon \left[\frac{\partial}{\partial \underline{x}_0} \cdot \underline{E}^{(1)} + 4\pi n_0 e \int f^{(1)} d\underline{v} \right] + \epsilon^2 \left[\frac{\partial}{\partial \underline{x}_0} \cdot \underline{E}^{(2)} + 4\pi n_0 e \int f^{(2)} d\underline{v} + \frac{\partial}{\partial \underline{x}_1} \cdot \underline{E}^{(1)} \right] = 0 , \quad (5)$$

$$\epsilon \left[\frac{\partial}{\partial \underline{x}_0} \cdot \underline{B}^{(1)} \right] + \epsilon^2 \left[\frac{\partial}{\partial \underline{x}_0} \cdot \underline{B}^{(2)} + \frac{\partial}{\partial \underline{x}_1} \cdot \underline{B}^{(1)} \right] = 0 , \quad (6)$$

$$\epsilon \left[\frac{\partial}{\partial \underline{x}_0} \times \underline{E}^{(1)} + \frac{1}{c} \frac{\partial \underline{B}^{(1)}}{\partial t_0} \right] + \epsilon^2 \left[\frac{\partial}{\partial \underline{x}_0} \times \underline{E}^{(2)} + \frac{1}{c} \frac{\partial \underline{B}^{(2)}}{\partial t_0} + \frac{\partial}{\partial \underline{x}_1} \times \underline{E}^{(1)} + \frac{1}{c} \frac{\partial \underline{B}^{(1)}}{\partial t_1} \right] = 0 , \quad (7)$$

and

$$\epsilon \left[\frac{\partial}{\partial \underline{x}_0} \times \underline{B}^{(1)} - \frac{1}{c} \frac{\partial \underline{E}^{(1)}}{\partial t_0} + \frac{4\pi n_0 e}{c} \int \underline{v} f^{(1)} d\underline{v} \right] + \epsilon^2 \left[\frac{\partial}{\partial \underline{x}_0} \times \underline{B}^{(2)} - \frac{1}{c} \frac{\partial \underline{E}^{(2)}}{\partial t_0} + \frac{4\pi n_0 e}{c} \int \underline{v} f^{(2)} d\underline{v} + \frac{\partial}{\partial \underline{x}_1} \times \underline{B}^{(1)} - \frac{1}{c} \frac{\partial \underline{E}^{(1)}}{\partial t_1} \right] = 0 . \quad (8)$$

We shall look for steady state solutions on the "fast" scales, and shall assume all functions of \underline{x}_0 and t_0 to be expanded in Fourier series with real wavenumbers and frequencies. Therefore we seek solutions of the form

$$\begin{pmatrix} f^{(\nu)} \\ \underline{E}^{(\nu)} \\ \underline{B}^{(\nu)} \end{pmatrix} = \frac{1}{VT} \sum_{\underline{k}, \omega} \begin{pmatrix} f_{\underline{k}\omega}^{(\nu)}(\underline{x}_1, t_1, \dots; \underline{\nu}) \\ \underline{E}_{\underline{k}\omega}^{(\nu)}(\underline{x}_1, t_1, \dots) \\ \underline{B}_{\underline{k}\omega}^{(1)}(\underline{x}_1, t_1, \dots) \end{pmatrix} e^{i(\underline{k} \cdot \underline{x}_0 - \omega t_0)}, \quad (9)$$

where $\nu = 1, 2, \dots$. This essentially means that $\underline{E}^{(\nu)}$ is undamped on the scales specified by \underline{x}_0 and t_0 . Any damping, which may occur, proceeds at a rate of $O(\epsilon)$, and is incorporated in the "slow" time and space scale dependence of the Fourier amplitudes.

From Eq. (4), to $O(\epsilon)$ and $O(\epsilon^2)$, for $\underline{k} \neq 0$ and $\omega \neq 0$ we obtain

$$f_{\underline{k}\omega}^{(1)} = \frac{e i}{m} \frac{\underline{E}_{\underline{k}\omega}^{(1)} \cdot \frac{\partial f^{(0)}}{\partial \underline{\nu}}}{\omega - \underline{k} \cdot \underline{\nu} + i \eta} \quad (10)$$

and

$$\begin{aligned}
 \underline{f}_{\underline{k}\omega}^{(2)} &= \frac{\underline{e}i}{m} \frac{\underline{E}_{\underline{k}\omega}^{(2)} \cdot \frac{\partial \underline{f}^{(0)}}{\partial \underline{v}}}{\omega - \underline{k} \cdot \underline{v} + i\eta} \\
 &+ \frac{1}{VT} \frac{\underline{e}i}{m} \sum_{\underline{k}', \omega'} \frac{(\underline{E}_{\underline{k}'\omega'}^{(1)} + \frac{1}{c} \underline{v} \times \underline{B}_{\underline{k}'\omega'}^{(1)}) \cdot \frac{\partial \underline{f}_{\underline{k}-\underline{k}', \omega-\omega'}^{(1)}}{\partial \underline{v}}}{\omega - \underline{k} \cdot \underline{v} + i\eta} \\
 &- i \frac{\left(\frac{\partial \underline{f}_{\underline{k}\omega}^{(1)}}{\partial t_1} + \underline{v} \cdot \frac{\partial \underline{f}_{\underline{k}\omega}^{(1)}}{\partial \underline{x}_1} \right)}{\omega - \underline{k} \cdot \underline{v} + i\eta}, \tag{11}
 \end{aligned}$$

where η is a small positive number.

It will sometimes be convenient to consider the longitudinal and transverse components of $\underline{E}_{\underline{k}\omega}^{(\nu)}$ separately. We shall denote them by the symbols $\underline{E}_{L\underline{k}\omega}^{(\nu)}$ and $\underline{E}_{T\underline{k}\omega}^{(\nu)}$, respectively.

Substituting Eqs. (10) and (11) into Eqs. (5) and (7), we obtain, for $\underline{k} \neq 0$ and $\omega \neq 0$,

$$\begin{aligned}
 &\epsilon D_L(\underline{k}, \omega) |\underline{E}_{L\underline{k}\omega}^{(1)}| + \epsilon^2 D_L(\underline{k}, \omega) |\underline{E}_{L\underline{k}\omega}^{(2)}| \\
 &= \epsilon^2 \left[\frac{1}{k} \frac{\partial}{\partial \underline{x}_1} \cdot \underline{E}_{\underline{k}\omega}^{(1)} + \frac{4\pi n_0 e}{k} \int \frac{\left(\frac{\partial \underline{f}_{\underline{k}\omega}^{(1)}}{\partial t_1} + \underline{v} \cdot \frac{\partial \underline{f}_{\underline{k}\omega}^{(1)}}{\partial \underline{x}_1} \right)}{\omega - \underline{k} \cdot \underline{v} + i\eta} d\underline{v} \right]
 \end{aligned}$$

$$- \frac{1}{VT} \frac{\omega^2}{k} \sum_{\underline{k}', \omega'} \left[\frac{(\underline{E}_{\underline{k}', \omega'}^{(1)} + \frac{1}{c} \underline{v} \times \underline{B}_{\underline{k}', \omega'}^{(1)}) \cdot \frac{\partial f_{\underline{k}-\underline{k}', \omega-\omega'}^{(1)}}{\partial \underline{v}}}{\omega - \underline{k} \cdot \underline{v} + i\eta} d\underline{v} \right], \quad (12)$$

where $D_L(\underline{k}, \omega) = 1 + \frac{\omega^2}{k^2} \int \frac{\underline{k} \cdot \frac{\partial f^{(0)}}{\partial \underline{v}}}{\omega - \underline{k} \cdot \underline{v} + i\eta} d\underline{v}$, the longitudinal dielectric function; and

$$\begin{aligned} & \epsilon \left[D_T(\underline{k}, \omega) \underline{E}_{T\underline{k}\omega}^{(1)} + D_L(\underline{k}, \omega) \underline{E}_{L\underline{k}\omega}^{(1)} \right] + \epsilon^2 \left[D_T(\underline{k}, \omega) \underline{E}_{T\underline{k}\omega}^{(2)} + D_L(\underline{k}, \omega) \underline{E}_{L\underline{k}\omega}^{(2)} \right. \\ & + \frac{1}{VT} \frac{\omega^2}{\omega} \sum_{\underline{k}', \omega'} \int \frac{\underline{v} (\underline{E}_{\underline{k}', \omega'}^{(1)} + \frac{1}{c} \underline{v} \times \underline{B}_{\underline{k}', \omega'}^{(1)}) \cdot \frac{\partial f_{\underline{k}-\underline{k}', \omega-\omega'}^{(1)}}{\partial \underline{v}}}{\omega - \underline{k} \cdot \underline{v} + i\eta} d\underline{v} \\ & - \frac{4\pi n_0 e}{\omega} \int \frac{\underline{v} \left(\frac{\partial f_{\underline{k}\omega}^{(1)}}{\partial t_1} + \underline{v} \cdot \frac{\partial f_{\underline{k}\omega}^{(1)}}{\partial \underline{x}_1} \right)}{\omega - \underline{k} \cdot \underline{v} + i\eta} d\underline{v} + \frac{i}{\omega} \left(1 + \frac{c^2 k^2}{\omega^2} \right) \frac{\partial \underline{E}_{T\underline{k}\omega}^{(1)}}{\partial t_1} \\ & + \frac{i}{\omega} \frac{\partial \underline{E}_{L\underline{k}\omega}^{(1)}}{\partial t_1} - \frac{ic^2}{\omega^2} \underline{k} \left(\frac{\partial}{\partial \underline{x}_1} \cdot \underline{E}_{T\underline{k}\omega}^{(1)} \right) + \frac{2ic^2}{\omega^2} (\underline{k} \cdot \frac{\partial}{\partial \underline{x}_1}) \underline{E}_{T\underline{k}\omega}^{(1)} \\ & \left. - \frac{ic^2}{\omega^2} \underline{k} \times \left(\frac{\partial}{\partial \underline{x}_1} \times \underline{E}_{L\underline{k}\omega}^{(1)} \right) \right] = 0, \quad (13) \end{aligned}$$

where $D_T(\underline{k}, \omega) = 1 - \frac{c^2 \underline{k}^2}{\omega^2} - \frac{\omega_p^2}{\omega} \int \frac{f^{(0)}}{\omega - \underline{k} \cdot \underline{v} + i\eta} d\underline{v}$, the transverse dielectric function.

Since the quiescent state of the system is field-free, we require that $\underline{E}_{\underline{k}=0, \omega=0}^{(1)} = 0$. Hence the $\underline{k} = 0$ and $\omega = 0$ component of Eq. (4) yields

$$\frac{\partial f^{(0)}}{\partial t_1} + \underline{v} \cdot \frac{\partial f^{(0)}}{\partial \underline{x}_1} = 0.$$

We shall take $\frac{\partial f^{(0)}}{\partial t_1} = \frac{\partial f^{(0)}}{\partial \underline{x}_1} = 0$. Therefore $f^{(0)} = f^{(0)}(|\underline{v}|, \underline{x}_2, t_2, \underline{x}_3, t_3, \dots)$.

C. SECULAR TERMS IN THE LONGITUDINAL AND TRANSVERSE FOURIER COMPONENTS

Eq. (12) shows that secular terms appear in the longitudinal fields when $D_L \sim 0(\epsilon)$. They can be eliminated by choosing the dependence of the $0(\epsilon)$ longitudinal Fourier components on \underline{x}_1 and t_1 such that the sum of the terms involving derivatives with respect to \underline{x}_1 and t_1 cancel the secular term.

To perform the two integrations on the right side of Eq. (12), we eliminate $f_{\underline{k}\omega}^{(1)}$ and $f_{\underline{k}-\underline{k}', \omega-\omega'}^{(1)}$ by means of Eqs. (10). We notice that the nonlinear term contains only the coupling of transverse components, there being

no terms involving the coupling of longitudinal components or the coupling of longitudinal with transverse components.

We also notice that the two integrals on the right side of Eq. (12) can be split up into two terms, a principal value integral of order ϵ^2 and a pole contribution which detailed estimate shows to be of order $\epsilon^{5/2}$. Since the pole contribution is formally of higher order it will henceforth be neglected. This neglect corresponds to the neglect of non-linear Landau damping¹⁰. Since the general effect of the non-linear Landau damping is the lessening of the total Landau damping from the linear value¹¹, by keeping the linear value we obtain a lower bound for mode coupling effects when Landau damping limits the size of the longitudinal components.

The principal value integrals are evaluated by expanding the integrands in a power series in $(V_o/(\omega/k))$. Only the lowest order non-vanishing terms are retained.

Having evaluated the integrals on the right side of Eq. (12) according to the procedure outlined above, we eliminate secular terms from Eq. (12) by requiring that for \underline{k} and ω for which $D_L(\underline{k}, \omega) \sim 0(\epsilon)$ the longitudinal component of the electric field satisfy the equation.

$$\begin{aligned} \frac{\partial E_{Lk\omega}^{(1)}}{\partial t_1} + \frac{V_o^2}{\omega} \left(\underline{k} \cdot \frac{\partial}{\partial \underline{x}_1} \right) E_{Lk\omega}^{(1)} - \frac{i\omega^3}{2\omega_p^2} D_L(\underline{k}, \omega) E_{Lk\omega}^{(1)} \\ = \frac{i\omega^3}{2k} I_{\underline{k}\omega} \end{aligned} \quad (14)$$

where

$$\begin{aligned}
 \underline{I}_{\underline{k}\omega} \approx & \frac{1}{VT} \frac{ie}{m\omega^2} \sum_{\underline{k}', \omega'} \frac{1}{\omega - \omega'} \left\{ \frac{2}{\omega} (\underline{k} \cdot \underline{E}_{\underline{k}', \omega'}^{(1)}) (\underline{k} \cdot \underline{E}_{\underline{k} - \underline{k}', \omega - \omega'}^{(1)}) \right. \\
 & + \frac{1}{\omega - \omega'} (\underline{k} \cdot \underline{E}_{\underline{k}', \omega'}^{(1)}) \left[(\underline{k} - \underline{k}') \cdot \underline{E}_{\underline{k} - \underline{k}', \omega - \omega'}^{(1)} \right] \\
 & \left. - \frac{1}{\omega'} \underline{k} \times (\underline{k}' \times \underline{E}_{\underline{k}', \omega'}^{(1)}) \cdot \underline{E}_{\underline{k} - \underline{k}', \omega - \omega'}^{(1)} \right\} \quad (15)
 \end{aligned}$$

the mode coupling term.

Eq. (13) shows that secular terms may arise in the transverse fields when $D_T \sim 0(\epsilon)$ or smaller. They can be eliminated by choosing the dependence of the transverse modes on \underline{x}_1 and t_1 such that the sum of the terms in Eq. (13) involving derivatives with respect to \underline{x}_1 and t_1 will cancel the secular terms. Proceeding in a manner analogous to that which led to Eq. (14), we eliminate secular terms from Eq. (13) by requiring that for \underline{k} and ω for which $D_T(\underline{k}, \omega) \sim 0(\epsilon)$ the transverse component of the electric field satisfy the equation

$$\frac{\partial \underline{E}_{T\underline{k}\omega}^{(1)}}{\partial t_1} + \frac{c^2}{\omega} (\underline{k} \cdot \frac{\partial}{\partial \underline{x}_1}) \underline{E}_{T\underline{k}\omega}^{(1)} = \frac{\omega^2}{2i} \left(\frac{\omega}{k} \underline{k} \cdot \underline{I}_{\underline{k}\omega} - \underline{J}_{\underline{k}\omega} \right) , \quad (16)$$

where

$$\begin{aligned}
 \underline{J}_{\underline{k}\omega} \cong & \frac{1}{VT} \frac{ie}{m} \sum_{\underline{k}', \omega'} \left\{ \frac{(\underline{k} \cdot \underline{E}_{\underline{k}-\underline{k}', \omega-\omega'}^{(1)}) \underline{E}_{\underline{k}', \omega'}^{(1)}}{\omega^2 (\omega-\omega')} \right. \\
 & + \frac{(\underline{k} \cdot \underline{E}_{\underline{k}', \omega'}^{(1)}) \underline{E}_{\underline{k}-\underline{k}', \omega-\omega'}^{(1)}}{\omega^2 (\omega-\omega')} + \frac{[(\underline{k}-\underline{k}') \cdot \underline{E}_{\underline{k}-\underline{k}', \omega-\omega'}^{(1)}] \underline{E}_{\underline{k}', \omega'}^{(1)}}{\omega (\omega-\omega')^2} \\
 & \left. + \frac{\underline{E}_{\underline{k}-\underline{k}', \omega-\omega'}^{(1)} \times (\underline{k}' \times \underline{E}_{\underline{k}', \omega'}^{(1)})}{\omega \omega' (\omega-\omega')} \right\} \quad (17)
 \end{aligned}$$

and $\underline{I}_{\underline{k}\omega}$ is given by Eq. (15).

D. CHOICE OF LOWEST ORDER NON-VANISHING MODES

Eq. (12) shows that when $D_L \sim 0(1)$, $\underline{E}_{L\underline{k}\omega}^{(1)} = 0$; $\underline{E}_{L\underline{k}\omega}^{(1)} \neq 0$ only for those waves for which $D_L \sim 0(\epsilon)$ or smaller. Eq. (13) shows that $\underline{E}_{T\underline{k}\omega}^{(1)} \neq 0$ only when $D_T \sim 0(\epsilon)$ or smaller.

For the work to follow we take the transverse $0(\epsilon)$ field $\underline{E}_T^{(1)}$ to be the superposition of two plane waves inside the plasma, *i.e.*

$$\begin{aligned} \underline{E}_T^{(1)}(\underline{x}_0, t_0, \underline{x}_1, t_1, \dots) = & \underline{E}_1(\underline{x}_1, t_1, \dots) \sin(\underline{k}_1 \cdot \underline{x}_0 - \omega_1 t_0 + \phi_1(\underline{x}_1, t_1, \dots)) \\ & + \underline{E}_2(\underline{x}_1, t_1, \dots) \sin(\underline{k}_2 \cdot \underline{x}_0 - \omega_2 t_0 + \phi_2(\underline{x}_1, t_1, \dots)), \end{aligned} \quad (18)$$

where $D_T(\underline{k}_1, \omega_1) = D_T(\underline{k}_2, \omega_2) \sim 0$, $D_L(\underline{k}_1 - \underline{k}_2, \omega_1 - \omega_2) \sim 0(\epsilon)$, and $(\underline{k}_1 \cdot \underline{E}_1)$ and $(\underline{k}_2 \cdot \underline{E}_2)$ are at most of $0(\epsilon)$. In (18) we assume that ϕ_1 is the same for both polarizations of \underline{E}_1 , and similarly for ϕ_2 . This assumption can be made consistently for all cases for which we solve explicitly.

We note that the form of (18) allows in principle for externally induced variations in the amplitudes and phases of the transverse modes.

Moreover, fourier-analyzing (18), according to Eq. (9), and substituting into Eq. (15), we see that $I_{\underline{k}\omega} \neq 0$ in Eq. (14) only for $\omega = \omega_1 - \omega_2$ and $\underline{k} = \underline{k}_1 - \underline{k}_2$, or $\omega = -(\omega_1 - \omega_2)$ and $\underline{k} = -(\underline{k}_1 - \underline{k}_2)$. Therefore, with the particular choice of $\underline{E}_T^{(1)}$ made in (18), the most general form of $\underline{E}_L^{(1)}(\underline{x}_0, t_0, \underline{x}_1, t_1, \dots)$, which is modified by transverse mode coupling, can be written in the form

$$\begin{aligned} \underline{E}_L^{(1)}(\underline{x}_0, t_0, \underline{x}_1, t_1, \dots) = & E_3(\underline{x}_1, t_1, \dots) \hat{k}_3 \cos(\underline{k}_3 \cdot \underline{x}_0 - \omega_3 t_0 + \phi_3) \\ & + E_4(\underline{x}_1, t_1, \dots) \hat{k}_3 \sin(\underline{k}_3 \cdot \underline{x}_0 - \omega_3 t_0 + \phi_3), \end{aligned} \quad (19)$$

where $\hat{k}_3 = \frac{k_3}{|\underline{k}_3|}$, $\underline{k}_3 = \underline{k}_1 - \underline{k}_2$, $\omega_3 = \omega_1 - \omega_2$, and $\phi_3 = \phi_1 - \phi_2$.

Multiplying Eq. (14) by $\frac{1}{VT} e^{i(\underline{k} \cdot \underline{x}_0 - \omega t_0)}$ and summing over \underline{k} and ω , we obtain the following two equations:

$$\begin{aligned} \frac{\partial E_3}{\partial t_1} + \frac{V_o^2}{\omega_3} \underline{k}_3 \cdot \frac{\partial E_3}{\partial \underline{x}_1} + \frac{1}{2} \gamma_L E_3 + E_4 \left(\frac{\partial \phi_3}{\partial t_1} + \frac{V_o^2}{\omega_3} \underline{k}_3 \cdot \frac{\partial \phi_3}{\partial \underline{x}_1} - \Delta \right) \\ = \frac{e\omega_p k_3}{4m\omega_1 \omega_2} (\underline{E}_1 \cdot \underline{E}_2) \end{aligned} \quad (20)$$

and

$$\frac{\partial E_4}{\partial t_1} + \frac{V_o^2}{\omega_3} \underline{k}_3 \cdot \frac{\partial E_4}{\partial \underline{x}_1} + \frac{1}{2} \gamma_L E_4 - E_3 \left(\frac{\partial \phi_3}{\partial t_1} + \frac{V_o^2}{\omega_3} \underline{k}_3 \cdot \frac{\partial \phi_3}{\partial \underline{x}_1} - \Delta \right) = 0, \quad (21)$$

where Δ and γ_L are defined such that $D_L(\underline{k}_3, \omega_3) = \frac{2\Delta}{\omega_p} + \frac{i\gamma_L}{\omega_p} \sim 0(\epsilon)$. γ_L is the linear Landau damping decrement, and Δ is the "detuning" factor to $0(\epsilon)$.

Eqs. (20) and (21) show that the longitudinal modes can be limited by linear Landau damping, by "detuning", and by the coupling between transverse modes (which enter through the "slow" derivatives).

Since the presence of a collision term would introduce, in the linear approximation, a damping decrement which plays a role analogous to that of the Landau damping decrement, in the work below we replace γ_L by $\gamma \left[\equiv (\gamma_L + \gamma_c) \right]$ where γ_c is the collisional damping decrement.

If we fourier-analyze both (18) and (19), according to Eq. (9), and substitute into Eq. (16), we obtain the following equations for \underline{E}_1 , \underline{E}_2 , ϕ_1 , and ϕ_2 :

$$\frac{\partial \underline{E}_1}{\partial \tau_1} + \frac{c^2}{\omega_1} \left(\underline{k}_1 \cdot \frac{\partial}{\partial \underline{x}_1} \right) \underline{E}_1 = \frac{ek_3}{4m\omega_2} E_3 \left[\frac{(\underline{k}_1 \cdot \underline{E}_2) \underline{k}_1}{k_1^2} - \underline{E}_2 \right], \quad (22)$$

$$\frac{\partial \underline{E}_2}{\partial \tau_1} + \frac{c^2}{\omega_2} \left(\underline{k}_2 \cdot \frac{\partial}{\partial \underline{x}_1} \right) \underline{E}_2 = - \frac{ek_3}{4m\omega_1} E_3 \left[\frac{(\underline{k}_2 \cdot \underline{E}_1) \underline{k}_2}{k_2^2} - \underline{E}_1 \right], \quad (23)$$

$$\underline{E}_1 \left(\frac{\partial \phi_1}{\partial \tau_1} + \frac{c^2}{\omega_1} \underline{k}_1 \cdot \frac{\partial \phi_1}{\partial \underline{x}_1} \right) = \frac{ek_3}{4m\omega_2} E_4 \left[\underline{E}_2 - \frac{(\underline{k}_1 \cdot \underline{E}_2) \underline{k}_1}{k_1^2} \right], \quad (24)$$

and

$$\underline{E}_2 \left(\frac{\partial \phi_2}{\partial \tau_1} + \frac{c^2}{\omega_2} \underline{k}_2 \cdot \frac{\partial \phi_2}{\partial \underline{x}_1} \right) = \frac{ek_3}{4m\omega_1} E_4 \left[\underline{E}_1 - \frac{(\underline{k}_2 \cdot \underline{E}_1) \underline{k}_2}{k_2^2} \right]. \quad (25)$$

III. DEPENDENCE OF MODES ON \underline{x}_1 and τ_1

A. CONSERVATION LAWS

Let us notice that $\frac{c^2}{\omega_1} \underline{k}_1$, $\frac{c^2}{\omega_2} \underline{k}_2$, and $\frac{v^2}{\omega_3} \underline{k}_3$ are, respectively, the group velocities of the two transverse modes and the longitudinal mode. We shall denote them by \underline{V}_{g1} , \underline{V}_{g2} , and \underline{V}_{g3} .

If we eliminate the terms containing ϕ_3 from Eq. (20) with the aid of Eq. (21), we can derive the following conservation equations from Eqs. (20), (22) and (23):

$$\left. \begin{aligned} & \frac{\partial}{\partial t_1} \left(\frac{E_1^2}{\omega_1} + \frac{E_2^2}{\omega_2} \right) + \frac{\partial}{\partial \underline{x}_1} \cdot \left(\frac{E_1^2}{\omega_1} \underline{v}_{g1} + \frac{E_2^2}{\omega_2} \underline{v}_{g2} \right) = 0 \\ \text{and} \\ & \frac{\partial}{\partial t_1} \left[\frac{E_1^2}{\omega_1} + \frac{(E_3^2 + E_4^2)}{\omega_3} \right] + \frac{\partial}{\partial \underline{x}_1} \cdot \left[\frac{E_1^2}{\omega_1} \underline{v}_{g1} + \frac{(E_3^2 + E_4^2)}{\omega_3} \underline{v}_{g3} \right] \\ & \quad + \gamma \frac{(E_3^2 + E_4^2)}{\omega_3} = 0 \end{aligned} \right\} (26)$$

We will now seek to make the physical meaning of Eqs. (26) more explicit. First we define U_1 to be the average with respect to \underline{x}_0 of the sum of the electric, the magnetic, and the mechanical field energy densities of the first transverse wave, U_2 to be the corresponding average for the second transverse wave, and U_3 to be the corresponding average for the longitudinal wave.

The average mechanical field energy density of the first wave is of the form $\left\langle \frac{1}{2} n_o m \underline{u}_1^2 \right\rangle_{av.}$ where \underline{u}_1 satisfies the equation

$$\frac{\partial \underline{u}_1}{\partial t_0} = - \frac{e}{m} \underline{E}_1 \sin(\underline{k}_1 \cdot \underline{x}_0 - \omega_1 t_0 + \phi_1) \quad .$$

Therefore $\left\langle \frac{1}{2} n_o m u_1^2 \right\rangle_{av.} = \frac{1}{2} \frac{\omega_p^2}{\omega_1^2} \frac{E_1^2}{8\pi}$, and $U_1 = \frac{1}{2} \left(\frac{E_1^2}{8\pi} + \frac{B_1^2}{8\pi} \right)$
 $+ \left\langle \frac{1}{2} n_o m u_1^2 \right\rangle_{av.} = \frac{E_1^2}{8\pi}$. Similarly $U_2 = \frac{E_2^2}{8\pi}$ and $U_3 = \frac{1}{2} \frac{(E_3^2 + E_4^2)}{8\pi}$
 $+ \left\langle \frac{1}{2} n_o m u_3^2 \right\rangle_{av.} = \frac{E_3^2 + E_4^2}{8\pi}$, where u_3 satisfies the equation

$$\frac{\partial u_3}{\partial t_o} = - \frac{e}{m} E_3 \cos(\underline{k}_3 \cdot \underline{x}_o - \omega_3 t_o + \phi_3) - \frac{e}{m} E_4 \sin(\underline{k}_3 \cdot \underline{x}_o - \omega_3 t_o + \phi_3) .$$

We now note that U_1 , U_2 , and U_3 satisfy a set of equations identical with Eqs. (26), with E_1^2 replaced by U_1 , E_2^2 by U_2 , and $(E_3^2 + E_4^2)$ by U_3 .

If we define $N_1 \left(\equiv \frac{U_1}{\hbar \omega_1} \right)$ to be the number density of quanta of both modes of polarizations of the first transverse modes; $N_2 \left(\equiv \frac{U_2}{\hbar \omega_2} \right)$, of the second transverse mode; and $N_3 \left(\equiv \frac{U_3}{\hbar \omega_3} \right)$, the number density of quanta of the longitudinal mode, then N_1 , N_2 and N_3 satisfy the conservation equations:

and

$$\left. \begin{aligned} \frac{\partial}{\partial t_1} (N_1 + N_2) + \frac{\partial}{\partial \underline{x}_1} \cdot (N_1 \underline{v}_{g1} + N_2 \underline{v}_{g2}) &= 0 \\ \frac{\partial}{\partial t_2} (N_1 + N_3) + \frac{\partial}{\partial \underline{x}_1} \cdot (N_1 \underline{v}_{g1} + N_3 \underline{v}_{g3}) &= -\gamma N_3 . \end{aligned} \right\} (27)$$

Eqs. (27) have a lucid interpretation in terms of a three-fluid model. We have three fluids of quanta streaming with fluid velocities \underline{V}_{g1} , \underline{V}_{g2} , and \underline{V}_{g3} , respectively. Because of the nonlinear coupling of modes, the quanta can be created and annihilated. Whenever a quantum of the higher-frequency transverse mode is annihilated, its energy is shared between a lower-frequency transverse quantum and a longitudinal quantum, which are simultaneously created; and vice versa, whenever a higher-frequency transverse quantum is created, a lower frequency transverse quantum and a longitudinal quantum are annihilated. At the same time, there is a dissipation process which destroys the longitudinal quanta and converts their energy into thermal energy.

Equations (27) differ from those derived by Sturrock^{6,12,13} in that they include the dissipation of the energy of the longitudinal mode by the processes of Landau and collisional damping.

B. RELATIVE IMPORTANCE OF THE LIMITING MECHANISMS

We present here expressions for estimating the relative strengths of the different mechanisms which limit the longitudinal mode. This is done by taking each mechanism separately, assuming that it alone is present, and estimating the strength of the longitudinal field. The most effective mechanism, in a particular situation, is the one which gives the smallest amplitude of the longitudinal mode.

Let k_T , ω_T , and E_T refer, respectively, to the characteristic wavenumber, frequency, and amplitude of the transverse field (remember that $\omega_T \sim ck_T$); and let k_L and E_L refer, respectively, to the wavenumber and amplitude of the longitudinal field. Let τ_N and λ_N be, respectively, the time and spatial scales of variation due to mode-mode coupling.

Eq. (20)-(25), when written in dimensional form, become

$$\left. \begin{aligned} \left(\frac{1}{\tau_N} + \frac{V_0^2 k_L}{\omega_p} \frac{1}{\lambda_N} + \frac{1}{2} \gamma + \Delta \right) E_L &= \frac{ek_L}{4m} \frac{\omega_p}{\omega_T^2} E_T^2 \\ \text{and} \\ \left(\frac{1}{\tau_N} + \frac{c}{\lambda_N} \right) E_L &= \frac{ek_L}{4m\omega_T} E_L \end{aligned} \right\} \quad (28)$$

For initial value problems

$$\frac{1}{\tau_N} = \frac{ek_L}{4m\omega_T} E_L = \frac{e}{4mc} \left(\frac{\omega_p}{\omega_T} \right) \left(\frac{k_L c}{\omega_p} \right) E_L \quad (29)$$

For boundary value problems

$$\frac{1}{\lambda_N} = \frac{ek_L}{4mc\omega_T} E_L = \frac{e}{4mc^2} \left(\frac{\omega_p}{\omega_T} \right) \left(\frac{ck_L}{\omega_p} \right) E_L \quad (30)$$

When "detuning" is dominant,

$$E_L = \frac{e}{4mc} \left(\frac{\omega_p}{\omega_T} \right)^2 \left(\frac{k_L c}{\omega_p} \right) \frac{E_T^2}{\Delta} \quad (31)$$

and when damping is dominant,

$$E_L = \frac{e}{2mc} \left(\frac{\omega_p}{\omega_T} \right)^2 \left(\frac{k_L c}{\omega_p} \right) \frac{E_T^2}{\gamma} \quad (32)$$

(We have $\gamma \equiv \gamma_L + \gamma_c$. The Landau damping decrement, γ_L , may be written in the form

$$\gamma_L \approx \pi c \left(\frac{2e^2}{mc^2} \right)^{\frac{1}{2}} n_0^{\frac{1}{2}} (ck_L \omega_p^{-1})^{-3} (v_0/c)^{-3} \times \\ \times \exp \left[-\frac{1}{2} (ck_L \omega_p^{-1})^{-2} (v_0/c)^{-2} \right] .$$

The collision damping decrement, γ_c , is given by¹⁴

$$\gamma_c \approx \frac{.714}{c^3} \left(\frac{4\pi e^2}{m} \right)^2 \left(\frac{v_0}{c} \right)^{-3} n_0 \ln \Lambda$$

where $\Lambda = 6\pi n_0 L_D^3$, with $L_D = (k_B T / 4\pi n_0 e^2)^{\frac{1}{2}}$. Here k_B is Boltzmann's constant and T is the electron "temperature".)

When resonant mode-coupling effects are dominant, and we are doing a pure initial value problem,

$$E_L = \left(\frac{\omega_p}{\omega_T} \right)^{\frac{1}{2}} E_T \quad . \quad (33)$$

When resonant mode-coupling effects are dominant , and we are doing a pure boundary value problem,

$$E_L = \left(\frac{c}{v_0} \right) \left(\frac{\omega_p}{\omega_T} \right)^{\frac{1}{2}} \left(\frac{\omega_p}{ck_L} \right)^{\frac{1}{2}} E_T \quad . \quad (34)$$

Eqs. (33) and (34) were obtained by neglecting γ and Δ in Eqs. (28) and then solving for E_L and τ_N or λ_N .

At this point we note that convective non-linearities may limit the longitudinal mode by detuning^{15,16}, with an amplitude which can be written as:

$$E_L \sim \left[(m\pi)^{1/2} c/2 \right]^{1/3} E_T^{2/3} (\omega_p/\omega_T)^{2/3} n_o^{1/6} / \alpha^{1/3} . \quad (35)$$

We have not derived this effect analytically since formally it is derived from second order terms in the expansion of (3).

Since

$$\frac{ck_L}{\omega_p} \sim \left[1 + 4(\omega_T/\omega_p)^2 \sin^2(\theta/2) \right]^{1/2}$$

(with θ the angle between the k vectors of the two transverse waves) by means of Eqs. (29)-(35) one concludes that λ_n and τ_N are least for $\omega_p/\omega_T \rightarrow 1$ approaching one from below. Under these conditions $ck_L/\omega_p \sim 1$. Then among the longitudinal limiting mechanisms, non-linear effects grow relatively stronger as E_T and V_o/c increase, and as n_o decreases.

We note that spatial resonant coupling limits the longitudinal mode less effectively than temporal resonant coupling. In fact, the ratio of the longitudinal field limited in time to that limited in space is proportional to $\left(\frac{V_o}{c}\right) \left(\frac{c}{\omega_p/k_L}\right)^{1/2}$. Hence at zero temperature, spatial coupling is incapable of limiting the longitudinal mode. This last fact can also be seen in the work of Montgomery⁷ for a cold plasma, which does not yield solutions when only the spatial nonlinearity due to resonant mode coupling is presumed to limit the longitudinal mode.

C. DETUNING IS DOMINANT

In this case we neglect the derivatives with respect to \underline{x}_1 and t_1 as well as the damping decrements in Eqs. (20) and (21). Since we fix the phase angles for the two polarizations of each transverse wave in (18) to be the same, we find that detuning alone is capable of limiting the longitudinal field only when no rotations of \underline{E}_1 and \underline{E}_2 occur. Therefore in this section we restrict ourselves to situations with $\frac{(\underline{k}_1 \cdot \underline{x} \underline{E}_2) \underline{x} \underline{k}_1}{k_1^2}$ parallel to \underline{E}_1 and $\frac{(\underline{k}_2 \cdot \underline{x} \underline{E}_1) \underline{x} \underline{k}_2}{k_2^2}$ parallel to \underline{E}_2 . The resultant equations leave us the option of doing a pure boundary value problem, a pure initial value problem, or a mixed problem.

If we do a pure boundary value problem, we obtain the following solution to Eqs. (20)-(25):

$$\left. \begin{aligned}
 \underline{E}_1(\underline{x}_1) &= \underline{E}_1(\underline{x}_1=0) \quad , \\
 \underline{E}_2(\underline{x}_1) &= \underline{E}_2(\underline{x}_1=0) \quad , \\
 E_3 &= 0 \quad , \\
 E_4 &= - \frac{e\omega_p |\underline{k}_3|}{4m\omega_1\omega_2} \frac{\underline{E}_1 \cdot \underline{E}_2}{\Delta} \quad , \\
 \phi_1 &= - \frac{e^2 k_3^2 \omega_p}{16m^2 c^2 \omega_2^2 k_1^2} \frac{(\underline{E}_1 \cdot \underline{E}_2)^2}{E_1^2 \Delta} \underline{k}_1 \cdot \underline{x}_1 + \phi_1(\underline{x}_1=0) \quad , \\
 \phi_2 &= - \frac{e^2 k_3^2 \omega_p}{16m^2 c^2 \omega_1^2 k_2^2} \frac{(\underline{E}_1 \cdot \underline{E}_2)^2}{E_2^2 \Delta} \underline{k}_2 \cdot \underline{x}_1 + \phi_2(\underline{x}_1=0) \quad .
 \end{aligned} \right\} (36)$$

and

If we take \underline{k}_1 parallel to \underline{k}_2 and \underline{E}_1 parallel to \underline{E}_2 , our results are essentially identical with those of Montgomery⁷.

Had we done a pure initial value problem instead of a pure boundary value problem, our results would have been:

$$\begin{aligned}
 \underline{E}_1(t_1) &= \underline{E}_1(t_1=0) \quad , \\
 \underline{E}_2(t_1) &= \underline{E}_2(t_1=0) \quad , \\
 \underline{E}_3 &= 0 \quad , \\
 \underline{E}_4 &= - \frac{e\omega_p |\underline{k}_3|}{4m\omega_1\omega_2} \frac{\underline{E}_1 \cdot \underline{E}_2}{\Delta} \quad , \\
 \phi_1 &= - \frac{e^2 k_3^2 \omega_p}{16m^2 \omega_1 \omega_2} \frac{(\underline{E}_1 \cdot \underline{E}_2)^2}{E_1^2 \Delta} t_1 + \phi_1(t_1=0) \quad , \\
 \text{and} \\
 \phi_2 &= - \frac{e^2 k_3^2 \omega_p}{16m^2 \omega_1 \omega_2} \frac{(\underline{E}_1 \cdot \underline{E}_2)^2}{E_2^2 \Delta} t_1 + \phi_2(t_1=0) \quad .
 \end{aligned} \tag{37}$$

We see from Eqs. (36) and (37) that the effects of mode-mode coupling can be interpreted to produce "wave-number shifts" in the case of a pure boundary value problem, and "frequency shifts" in the case of a pure initial value problem.

D. DAMPING IS DOMINANT

In this case we neglect the "slow" derivatives and the "detuning" terms in Eqs. (20) and (21). We immediately obtain that $E_4 = 0$ and

$$E_3 = \frac{e\omega_p k_3}{2m\omega_1\omega_2} \frac{(\underline{E}_1 \cdot \underline{E}_2)}{\gamma} \quad 17.$$

Because of Eqs. (24) and (25), ϕ_1 and ϕ_2 may be taken to be constants.

To simplify the mathematics, let us take \underline{k}_1 parallel to \underline{k}_2 and pointing in the direction of the positive x-axis, and let us do a one-dimensional problem. We again have the option of doing an initial value or a boundary value problem; we shall do a boundary value problem. Let $\underline{E}_{10} \equiv \underline{E}_1(x_1=0)$ and $\underline{E}_{20} \equiv \underline{E}_2(x_1=0)$. Two cases will be treated: (1) \underline{E}_{10} parallel to \underline{E}_{20} , and (2) \underline{E}_{10} not parallel to \underline{E}_{20} .

Case (1):

In this case there is no rotation of the directions of polarization.

From Eqs. (20), (22), (23) and (26) we obtain the equations

$$\left. \begin{aligned} \frac{\partial E_1}{\partial x_1} &= - \frac{e^2 k_3^2 \omega_p}{8m^2 c^2 \omega_2^2 k_1} \frac{E_1 E_2}{\gamma}, \\ \frac{\partial E_2}{\partial x_1} &= \frac{e^2 k_3^2 \omega_p}{8m^2 c^2 \omega_1^2 k_2} \frac{E_1 E_2}{\gamma}, \end{aligned} \right\} \quad (38)$$

and

$$\frac{k_1 E_1^2}{\omega_1^2} + \frac{k_2 E_2^2}{\omega_2^2} = \frac{k_1 E_{10}^2}{\omega_1^2} + \frac{k_2 E_{20}^2}{\omega_2^2}.$$

Eqs. (38) can be easily integrated, and have the solution

$$\frac{E_1^2}{E_{10}^2} = \frac{\frac{k_1 E_{10}^2}{\omega_1^2} + \frac{k_2 E_{20}^2}{\omega_2^2}}{\frac{k_1 E_{10}^2}{\omega_1^2} + \frac{k_2 E_{20}^2}{\omega_2^2} e^{2\Lambda x_1}}$$

and

$$\frac{E_2^2}{E_{20}^2} = \frac{\frac{k_1 E_{10}^2}{\omega_1^2} + \frac{k_2 E_{20}^2}{\omega_2^2}}{\frac{k_1 E_{10}^2}{\omega_1^2} e^{-2\Lambda x_1} + \frac{k_2 E_{20}^2}{\omega_2^2}}$$

(39)

where $\Lambda = \frac{e^2 \omega_p k_3^2}{8m^2 c^2 k_1 k_2 \gamma} \left(\frac{k_1 E_{10}^2}{\omega_1^2} + \frac{k_2 E_{20}^2}{\omega_2^2} \right)$.

The solution (39) shows that, as $x \rightarrow \infty$, the higher-frequency transverse mode - and hence the longitudinal mode - is completely attenuated, and only the lower-frequency transverse mode remains.

Case (2):

Here we select \underline{E}_{10} and \underline{E}_{20} so that they form some arbitrary angle.

It is convenient to rewrite Eqs. (22) and (23) such that multiplicative constants do not appear explicitly. For this purpose, we define two vectors,

\underline{u} and \underline{v} , in the following way:

$$\underline{u} = \left(\frac{e^2 \omega_p k_3^2}{8m^2 c^2 \omega_1^2 k_2 \gamma} \right)^{\frac{1}{2}} \underline{E}_1, \quad \text{and} \quad \underline{v} = \left(\frac{e^2 \omega_p k_3^2}{8m^2 c^2 \omega_2^2 k_1 \gamma} \right)^{\frac{1}{2}} \underline{E}_2 .$$

Eqs. (20)-(23) can, therefore, be written in the form

$$\left. \begin{aligned} \frac{\partial \underline{u}}{\partial x_1} &= - \underline{v}(\underline{u} \cdot \underline{v}) \\ \text{and} \\ \frac{\partial \underline{v}}{\partial x_1} &= \underline{u}(\underline{u} \cdot \underline{v}) . \end{aligned} \right\} \quad (40)$$

Eqs. (40) yield the conservation equations

$$\left. \begin{aligned} u_y^2 + v_y^2 &= u_{y0}^2 + v_{y0}^2 \\ u_z^2 + v_z^2 &= u_{z0}^2 + v_{z0}^2 . \end{aligned} \right\} \quad (41)$$

The form of Eqs. (41) suggests that we parametrize Eqs. (40) in the following manner:

$$\left. \begin{aligned} \text{Let } u_y &= (u_{y0}^2 + v_{y0}^2)^{\frac{1}{2}} \cos \theta , \\ v_y &= (u_{y0}^2 + v_{y0}^2)^{\frac{1}{2}} \sin \theta , \\ u_z &= (u_{z0}^2 + v_{z0}^2)^{\frac{1}{2}} \cos \theta' , \\ \text{and} \\ v_z &= (u_{z0}^2 + v_{z0}^2)^{\frac{1}{2}} \sin \theta' . \end{aligned} \right\} \quad (42)$$

Evaluating $\frac{du}{du_z}$ from Eqs. (40), then from Eqs. (42), then equating the two expressions and integrating, we obtain

$$\theta' = \theta + \beta \quad , \quad (43)$$

where $\beta = \theta'_0 - \theta_0$, θ'_0 and θ_0 being the values of θ' and θ , respectively, at $x_1 = 0$. Defining three new quantities, ϕ , Λ , and α by

$$\left. \begin{aligned} \phi &\equiv 2\theta + \alpha + \pi \quad , \\ \Lambda &\equiv \left\{ \left[(u_{y_0}^2 + v_{y_0}^2) + (u_{z_0}^2 + v_{z_0}^2) \cos^2 \beta \right]^2 + (u_{z_0}^2 + v_{z_0}^2)^2 \sin^2 2\beta \right\}^{1/2} \quad , \\ \cos \alpha &\equiv \frac{(u_{y_0}^2 + v_{y_0}^2) + (u_{z_0}^2 + v_{z_0}^2) \cos 2\beta}{\Lambda} \quad , \\ \sin \alpha &\equiv \frac{(u_{z_0}^2 + v_{z_0}^2) \sin 2\beta}{\Lambda} \quad , \end{aligned} \right\} \quad (44)$$

and

Eqs. (40) reduce to only one equation,

$$\frac{\partial \phi}{\partial x_1} = -\Lambda \sin \phi \quad . \quad (45)$$

Eq. (45) has the solution $\tan \frac{\phi}{2} = e^{-\Lambda x_1} \tan \frac{\phi(x_1=0)}{2}$. But this implies that, when $x \rightarrow \infty$, $\underline{u} \cdot \underline{v} = -\frac{1}{2} \sin \phi = 0$, and \underline{E}_1 and \underline{E}_2 become perpendicular to one another. The asymptotic values of the fields follow from (42) and (44).

We see from cases (1) and (2) that the characteristic length for the higher frequency transverse wave being damped out, or the polarizations of the two transverse waves rotating until they are perpendicular, is $1/\Lambda$. Had we done a pure initial value problem, we would have obtained an analogous behavior of the waves in time, with the characteristic time $1/\Lambda$, where Λ is given by Eq. (44), but \underline{u} and \underline{v} are given by

$$\underline{u} = \left(\frac{e^2 \omega_p k_3^2}{8m^2 \omega_1 \omega_2 \gamma} \right)^{1/2} \underline{E}_1$$

and

$$\underline{v} = \left(\frac{e^2 \omega_p k_3^2}{8m^2 \omega_1 \omega_2 \gamma} \right)^{1/2} \underline{E}_2$$

E. RESONANT MODE COUPLING

In this case we neglect the detuning and the damping terms in Eqs. (20) and (21).

It can be shown that the fields are limited in time and in space. For example, in the case of a pure initial value problem, we obtain from Eqs. (27) that $(N_1(t_1) + N_2(t_1)) = (N_1(t_1=0) + N_2(t_1=0))$ and $(N_1(t_1) + N_3(t_1)) = (N_1(t_1=0) + N_3(t_1=0))$. Since N_1 , N_2 , and N_3 are finite at $t_1 = 0$, they must be finite for $t_1 > 0$.

In the case of a pure boundary value problem the argument is more difficult. However, if we restrict ourselves to one-dimensional situations, with \underline{k}_1 , \underline{k}_2 and \underline{k}_3 parallel to one another, we obtain from Eqs. (27) that $(N_1(x_1)V_{g1} + N_2(x_1)V_{g2}) = (N_1(x_1=0)V_{g1} + N_2(x_1=0)V_{g2})$ and $(N_1(x_1)V_{g1} + N_3(x_1)V_{g3}) = (N_1(x_1=0)V_{g1} + N_3(x_1=0)V_{g3})$. Since N_1 , N_2 and N_3 are finite at $x_1 = 0$, they must be finite for $x_1 > 0$.

We shall now do several pure boundary value problems. We shall assume that $E_3 = E_4 = 0$ on the boundary. It can be seen from Eqs. (20)-(25) that only E_3 can be built up from zero as we move away from the boundary, while E_4 remains zero. Hence ϕ_1 and ϕ_2 remain constant inside the plasma, and we have to solve only Eqs. (20), (22) and (23) for each boundary value problem.

To simplify the mathematics, let us define new field variable \underline{u} , \underline{v} , and w by

$$\left. \begin{aligned} \underline{u} &\equiv \left(\frac{e^2 \omega_p^2 k_3}{16m^2 c^2 \omega_1^2 k_2 V_o^2} \right)^{1/2} E_1, \\ \underline{v} &\equiv \left(\frac{e^2 \omega_p^2 k_3}{16m^2 c^2 \omega_2^2 k_1 V_o^2} \right)^{1/2} E_2, \\ \text{and} \\ w &\equiv \left(\frac{e^2 k_3^2}{16m^2 c^4 k_1 k_2} \right)^{1/2} E_3. \end{aligned} \right\} \quad (46)$$

Let us also take \underline{k}_1 and \underline{k}_2 to be parallel or "almost" parallel to each other and to point in the positive x-direction. Let \underline{u}_0 and \underline{v}_0 be the constant values of \underline{u} and \underline{v} specified on the plane $x = 0$. Eqs. (20), (22) and (23) then become

$$\left. \begin{aligned} \frac{\partial \underline{u}}{\partial x_1} &= -w \underline{v} , \\ \frac{\partial \underline{v}}{\partial x_1} &= w \underline{u} , \end{aligned} \right\} \quad (47)$$

and

$$\frac{k_3}{k_3} \cdot \frac{\partial w}{\partial x_1} = \underline{u} \cdot \underline{v} .$$

Eqs. (47) yield the conservation equations (41). They can be parametrized by means of Eqs. (42), with the relationship between θ and θ' being given by Eq. (43). If we also make use of the definitions (44), Eqs. (47) will reduce to the form

$$\left. \begin{aligned} \frac{\partial \phi}{\partial x_1} &= 2w \\ \frac{k_3}{k_3} \cdot \frac{\partial w}{\partial x_1} + \frac{1}{2} \Lambda \sin \phi &= 0 . \end{aligned} \right\} \quad (48)$$

Case (1):

\underline{k}_3 is parallel to \underline{k}_1 and \underline{k}_2 . We let $w(x_1=0) = 0$ and $\theta_0 \equiv \theta(x_1 = 0)$. In this case Eqs. (48) yield the equation

$$\frac{\partial^2 \phi}{\partial x_1^2} + \Lambda \sin \phi = 0 \quad . \quad (49)$$

Eq. (49) can be integrated to give

$$\int_{\frac{\phi(x_1=0)}{2}}^{\phi/2} \frac{d(\phi/2)}{\sqrt{\sin^2 \frac{1}{2} \phi(x_1=0) - \sin^2 \frac{1}{2} \phi}} = \pm \sqrt{\Lambda} x_1 .$$

If we define the number k , with $|k| < 1$, by the equation $k^2 = \sin^2 \frac{1}{2} \phi(x_1=0) = \cos^2(\theta_0 + \frac{\alpha}{2})$, then, by means of the substitution

$\sin \phi = \frac{1}{k} \sin \frac{\phi}{2}$, the integration of Eq. (49) can be written in the form

$$\int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \pm \sqrt{\Lambda} x_1 + \kappa \quad \text{where} \quad \kappa = \int_0^{\frac{1}{2}\pi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}, \quad \text{an}$$

elliptic integral of the first kind¹⁸.

Therefore \underline{u} , \underline{v} , and w can be expressed in terms of elliptic functions:¹⁸

$$u_y = (u_{y0}^2 + v_{y0}^2)^{1/2} \left[k \cos \frac{\alpha}{2} \operatorname{sn}(\sqrt{\Lambda} x_1 + \kappa, k) + \sin \frac{\alpha}{2} \operatorname{dn}(\sqrt{\Lambda} x_1 + \kappa, k) \right],$$

$$v_y = (u_{y0}^2 + v_{y0}^2)^{1/2} \left[\cos \frac{\alpha}{2} \operatorname{dn}(\sqrt{\Lambda} x_1 + \kappa, k) - k \sin \frac{\alpha}{2} \operatorname{sn}(\sqrt{\Lambda} x_1 + \kappa, k) \right],$$

$$u_z = (u_{z0}^2 + v_{z0}^2)^{1/2} \left[k \cos(\frac{\alpha}{2} - \beta) \operatorname{sn}(\sqrt{\Lambda} x_1 + \kappa, k) + \sin(\frac{\alpha}{2} - \beta) \operatorname{dn}(\sqrt{\Lambda} x_1 + \kappa, k) \right],$$

$$v_z = (u_{z0}^2 + v_{z0}^2)^{1/2} \left[\cos(\frac{\alpha}{2} - \beta) \operatorname{dn}(\sqrt{\Lambda} x_1 + \kappa, k) - k \sin(\frac{\alpha}{2} - \beta) \operatorname{sn}(\sqrt{\Lambda} x_1 + \kappa, k) \right],$$

and

$$w = -k\sqrt{\Lambda} \operatorname{cn}(\sqrt{\Lambda} x_1 + \kappa, k) .$$

(50)

Eqs.(50) show that the amplitudes of the modes are oscillatory functions of x_1 .

In the case of a one-dimensional pure initial value problem we would obtain results which are identical with Eq. (50); but with x_1 replaced by t_1 ; the \underline{u} , \underline{v} , and w defined by

$$\underline{u} = \left(\frac{e^2 \omega_p k_3^2}{16m^2 \omega_1 \omega_2} \right)^{1/2} \underline{E}_1 ,$$

$$\underline{v} = \left(\frac{e^2 \omega_p k_3^2}{16m^2 \omega_1 \omega_2} \right)^{1/2} \underline{E}_2 ,$$

$$w = \left(\frac{e^2 k_3^2}{16m^2 \omega_1 \omega_2} \right)^{1/2} E_3 ;$$

and the \underline{u}_0 and \underline{v}_0 being the values of \underline{u} and \underline{v} at $t = 0$. ($E_3(t_1=0) = 0$).

Case (2):

We consider the special two dimensional situation of Diagram 1.

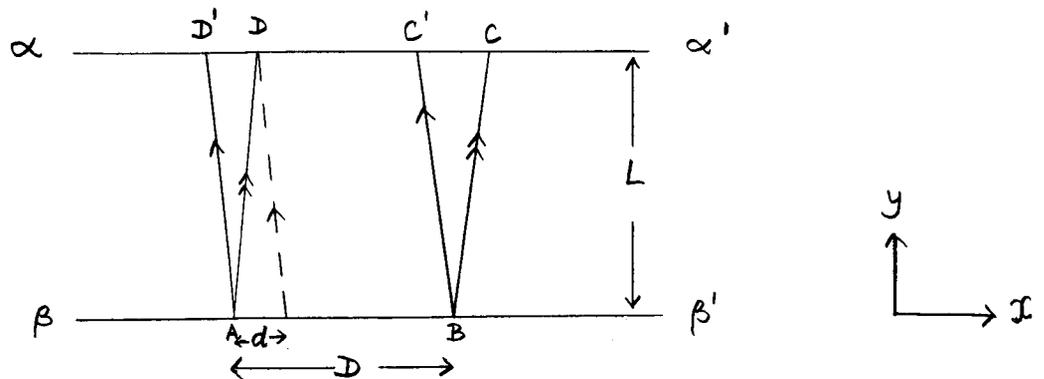


Diagram 1.

The resonant plasma region boundaries on the horizontal lines $\alpha\alpha'$ and $\beta\beta'$. The lines AD' and BC' , both parallel to $\frac{k_1}{|k_1|}$, denote the bounds of the region within which the mode (ω_1, \vec{k}_1) is localized. The lines AD and BC bear a similar relation with respect to the mode (ω_2, \vec{k}_2) . We assume

$$\frac{|k_1 \times k_2|}{|k_1||k_2|} \ll 1, \quad (51)$$

$$\frac{k_1 \cdot k_3}{|k_1||k_3|} \ll 1, \quad (52)$$

$$L \frac{\frac{\partial E_1}{\partial y}(x=d)}{E_1(y=0)} \ll 1, \quad (53)$$

$$L \frac{\frac{\partial E_1}{\partial y}(x=0)}{E_1(y=0)} \geq 1, \quad (54)$$

On using (20) and (22), conditions (53) and (54) become

$$\left(\frac{e \omega_p k_3}{4m\omega_1\omega_2 v_o^2} \frac{\omega_3}{k_3} \right) \left(\frac{ek_3}{4m\omega_3 c^2} \frac{\omega_1}{k_1} \right) E_1 E_2 L \frac{|k_1 \times k_2|}{|k_1||k_2|} \ll 1, \quad (55)$$

$$\left(\frac{e \omega_p k_3}{4m\omega_1\omega_2 v_o^2} \frac{\omega_3}{k_3} \right) \left(\frac{ek_3}{4m\omega_3 c^2} \frac{\omega_1}{k_1} \right) E_1 E_2 D \geq 1. \quad (56)$$

Inequality (56) insures the occurrence of mode coupling effects in the volume of diagram 1.

In view of (56) we may look for solutions to Eqs. (48) of the form $\phi = \phi(\xi)$, where $\xi \equiv (4\Lambda x_1 y_1)^{1/2}$. With this substitution, Eqs. (48) becomes

$$\frac{d^2 \phi}{d\xi^2} + \frac{1}{\xi} \frac{d\phi}{d\xi} + \sin \phi = 0 . \quad (57)$$

Eq. (57) is difficult to solve. However, we can get some idea of the behavior by considering small values of ϕ_0 ($\equiv \phi(x_1=0, y_1=0)$). Then $\sin \phi \approx \phi$, and the solution of Eq. (57) is of the form $\phi = \phi_0 J_0(\xi)$ for $\xi \ll (\phi_0)^{-2}$. Here $J_0(\xi)$ is the Bessel function of the first kind of order zero, so that ϕ is an oscillating function of ξ which approaches zero for $\xi \gg 1$.

The restriction that ϕ be small implies that if \underline{u}_0 is parallel to \underline{v}_0 , $|\underline{u}_0|$ is taken to be small; however, when \underline{u}_0 is not parallel to \underline{v}_0 , \underline{u}_0 and \underline{v}_0 can have arbitrary magnitudes, but have to be almost perpendicular.

The asymptotic limit of ϕ implies that $w(\xi \rightarrow \infty) = 0$; also $\underline{u}(\xi \rightarrow \infty) = 0$ when \underline{u}_0 is parallel to \underline{v}_0 , and $\underline{u} \cdot \underline{v} = 0$ when \underline{u}_0 is not parallel to \underline{v}_0 .

CONCLUSIONS

We have established the connection between Sturrock's mode-coupling equations⁶ which were derived by means of a derivative expansion technique, and the quasi-linear solution of the non-linear Vlasov equation and Maxwell's equations, generalizing the former to include dissipation, finite temperature and spatial variation effects. For the system of two transverse modes and the resulting driven longitudinal mode we have determined and treated non-linear frequency conversion and polarization rotation effects. We note that in the absence of detuning and damping limitations, for problems involving a single coordinate there is a characteristic evolution which is basically oscillatory in that it is describable in terms of elliptic functions; whereas even for a relatively simple two-dimensional problem the evolution of the transverse modes is characterized by an oscillation of decreasing magnitude which tends towards the same limit as when damping alone controls the longitudinal oscillations. From an experimental viewpoint the mode coupling effects are optimized by the choice of (ω_p/ω_T) of order unity.

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